09.08 Power Series

1. The objectives of this section are to:
   - Understand the definition of a power series.
   - Find the radius and interval of convergence of a power series.
   - Determine the endpoint convergence of a power series.
   - Differentiate and integrate a power series

2. **Definition of Power Series**: If x is a variable, then an infinite series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + ... + a_n(x - c)^n + ...$ is called a **power series centered at c**, where c is a constant.
   Note: To simplify the notation for power series, we agree that $(x - c)^0 = 1$, even if x = c.

3. A power series in x can be viewed as a function of x. $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ where the domain of f is the set of all x for which the series converges.

4. Determination of the domain of a power series is the primary concern in this section.

5. Of course, every power series converges at its center c because $f(c) = \sum_{n=0}^{\infty} a_n(c - c)^n = a_0(1) + 0 + 0 + ... + 0 + ... = a_0$. So, c always lies in the domain of f. The following important theorem states that the domain of a power series can take three basic forms: a single point, an interval centered at c, or the entire real line.

6. **Convergence of a Power Series**: For a power series centered at c, precisely one of the following is true.
   1. The series converges only at c.
   2. There exists a real number R > 0 such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
   3. The series converges absolutely for all x.

   The number R is the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is R = 0, and if the series converges for all x, the radius of converges $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

7. To determine the **radius of convergence** of a power series, use the **Ratio Test**.

8. To determine the **interval of convergence** test the end points about the radius for convergence.

9. Find the radius of convergence of $\sum_{n=0}^{\infty} 3(x - 2)^n$
   Solution: For $x \neq 2$, let $u_n = 3(x - 2)^n$. Then
   $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty} \left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right| = \lim_{n \to \infty} |x-2| = |x-2|$. By the Ratio Test, the series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$. Therefore the radius of convergence is R = 1.

10. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Solution: Let $u_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Then

$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2n+3} (2n+1)!}{(-1)^n x^{2n+1} (2n+3)!} \right| = \lim_{n \to \infty} \frac{x^2}{(2n+3)(2n+2)}$. For any fixed value of x, this limit is 0. So by the Ratio Test, the series converges for all x, and the radius of convergence is $R = \infty$.

11. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$

Solution: Let $u_n = \frac{x^n}{n}$. Then

$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}\, n}{x^n\, (n+1)} \right| = \lim_{n \to \infty} \left| \frac{xn}{n+1} \right| = |x| < 1$. By the Ratio Test, the radius of convergence is R = 1. Since the series is centered at 0, it converges in the interval (-1, 1). This is not necessarily the interval of convergence. To determine this, we must test for convergence at each endpoint.
When x = 1, we get the divergent harmonic series.
When x = -1, we get the convergent alternating harmonic series.
The interval of convergence is [-1, 1).

12. Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$

Solution: Let $u_n = \frac{(-1)^n (x+1)^n}{2^n}$. Then

$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (x+1)^{n+1}\, 2^n}{(-1)^n (x+1)^n\, 2^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{x+1}{2} \right| = \left| \frac{x+1}{2} \right| < 1 \rightarrow |x+1| < 2$. By the Ratio Test, the radius of convergence is 2. The series is centered about -1, so we must test each endpoint.

When x = -3, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty} 1$ which is divergent

When x = 1, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n(2)^n}{2^n} = \sum_{n=0}^{\infty} (-1^n)$ which is divergent

The interval of convergence is (-3, 1) .

13. **Properties of Functions Defined by Power Series**: If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + ...$$

has a radius of convergence R > 0, then, on the interval (c - R, c + R), f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $$f'(x) = \sum_{n=1}^{\infty} na_n(x - c)^{n-1} = a_1 + 2a_2(x - c) + 3a_3(x - c)^3 + ...$$

2. $$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1} = C + a_0(x - c) + a_1 \frac{(x - c)^2}{2} + a_2 \frac{(x - c)^3}{3} + ...$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of the behavior at the endpoints.

14. Find the derivative of $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

Solution: $f'(x) = 1 + (2)\frac{x}{2} + (3)\frac{x^2}{3!} + (4)\frac{x^3}{4!} + ... = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... = f(x)$

Recall that in this example, f(x) is the power series for e^x.

15. Given $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + ...$ Find the intervals of convergence of $\int f(x)\,dx$, f(x), and f '(x).

Solution: $f'(x) = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + ...$

$$\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n + 1)} = C + \frac{x^2}{1 \bullet 2} + \frac{x^3}{2 \bullet 3} + \frac{x^4}{3 \bullet 4} + ...$$

Ratio Test: $\lim_{n \to \infty} \left| \frac{x^{n+1}\, n}{x^n\,(n+1)} \right| = \lim_{n \to \infty} \left| \frac{xn}{n+1} \right| = |x| < 1$. The Radius of convergence is R = 1.

At endpoint x = -1: $\int f(x)\,dx$ converges, f(x) converges, f '(x) diverges.

At endpoint x = 1: $\int f(x)\,dx$ converges, f(x) diverges, f '(x) diverges.

The Interval of convergence for $\int f(x)\,dx$ is [-1, 1].

The Interval of convergence for f(x) is [-1, 1).

The Interval of convergence for f '(x) is (-1, 1).

AP Calculus BC 2 Assignment 211 Monday, February 23, 2015 Hour Name ______________________________

1. $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n+1}$ Find the radius of convergence.

2. $\sum_{n=0}^{\infty}\frac{(2x)^n}{n^2}$ Find the radius of convergence.

3. $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$ Find the interval of convergence.

4. $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$ Find the interval of convergence.

5. $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ Find the interval of convergence.

6. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$ Find the interval of convergence.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$ Find the interval of convergence.

8. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-5)^n}{n 5^n}$ Find the interval of convergence.

9. $\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$ Find the interval of convergence.

10. f(x) = $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ (x-5)^n}{n\ 5^n}$ Find the interval of convergence of f '(x).

11. f(x) = $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ (x-5)^n}{n\, 5^n}$ Find the interval of convergence of $\int f(x)\, dx$

12. f(x) = $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ (x-5)^n}{n\, 5^n}$ Find the interval of convergence of f ''(x).