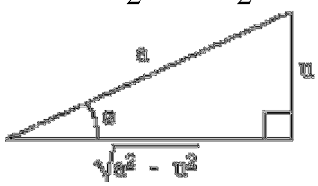
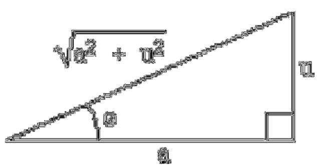
Trigonometric Substitution Notes (Trig Substitution)

Trigonometric Substitution: a > 0

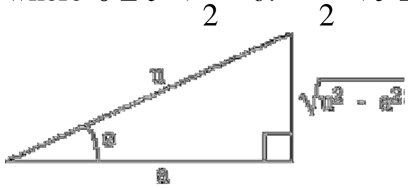
a. For integrals involving $\sqrt{a^2-u^2}$, let u = a sinθ. Then $\sqrt{a^2-u^2} = \sqrt{a^2-a^2\sin^2\theta} = a\cos\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.



b. For integrals involving $\sqrt{a^2+u^2}$, let u = a tanθ. Then $\sqrt{a^2+u^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sec\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.



c. For integrals involving $\sqrt{u^2-a^2}$, let u = a secθ. Then $\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\theta-a^2} = \pm a\tan\theta$, where $0 \le \theta < \frac{\pi}{2}$ *or* $\frac{\pi}{2} < \theta \le \pi$. Use the positive value if u > a and the negative value if u < -a.



Examples:

1. Find $\int \frac{dx}{x^2\sqrt{9-x^2}}$

Solution: Use the substitution x = 3 sin$\theta \rightarrow$dx = 3 cos$\theta\, d\theta$ $\int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}} = \int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\, 3\cos\theta}$

$= \frac{1}{9}\int \csc^2\theta\, d\theta = -\frac{1}{9}\cot\theta\, d\theta + C$ Since $\sin\theta = \frac{x}{3} = \frac{opp}{hyp}$, then adj = $\sqrt{9-x^2}$ So we get

$-\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C = -\frac{\sqrt{9-x^2}}{9x} + C$

2. Find $\int \frac{dx}{\sqrt{4x^2+1}}$

Solution: Use the substitution 2x = tanθ $\rightarrow$dx $= \frac{1}{2}\sec^2\theta\, d\theta \rightarrow \int \frac{\sec^2\theta\, d\theta}{2\sqrt{\tan^2\theta+1}} = \frac{1}{2}\int \sec\theta\, d\theta$

$= \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$ Since $\tan\theta = \frac{2x}{1} = \frac{opp}{adj}$, then hyp = $\sqrt{4x^2+1}$ So we get

$\frac{1}{2}\ln\left|\sqrt{4x^2+1}+2x+C\right|$

3. Find $\int \frac{dx}{\left(x^2+1\right)^{3/2}}$

Solution: Use the substitution x = $\tan\theta \rightarrow$ dx = $\sec^2\theta$ dθ $\rightarrow \int \frac{\sec^2\theta\, d\theta}{\left(\tan^2\theta+1\right)^{3/2}} = \int \frac{d\theta}{\sec\theta} = \cos\theta\, d\theta$

= $\sin\theta$ + C Since $\tan\theta$ = $\frac{x}{1} = \frac{opp}{adj}$, then hyp = $\sqrt{x^2+1}$ So we get $\frac{x}{\sqrt{x^2+1}} + C$

4. Find $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\, dx$

Solution: Use the substitution x = $\sqrt{3}\sec\theta \rightarrow$ dx = $\sqrt{3}\sec\theta\ \tan\theta\ \ d\theta \rightarrow$ When x = $\sqrt{3}$, θ = 0.

When x = 2, θ = $\frac{\pi}{6} \rightarrow \int_0^{\pi/6} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta} \sec\theta \tan\theta\, d\theta$ = $\sqrt{3}\int_0^{\pi/6} \tan^2\theta\, d\theta$ = $\sqrt{3}\int_0^{\pi/6} \left(\sec^2\theta - 1\right)^2 d\theta$

= $\sqrt{3}\left[\tan\theta - \theta\right]_0^{\pi/6}$ = $\sqrt{3}\left[\frac{\sqrt{3}}{3} - \frac{\pi}{6}\right]$ = 1 - $\frac{\sqrt{3}\pi}{6}$ = 0.0931

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Use Trigonometric Substitution if possible.

1. $\int \frac{x}{\sqrt{9-x^2}}\,dx =$

2. $\int \sqrt{16-4x^2}\,dx =$

3. $\int \frac{1}{\sqrt{x^2-9}}\,dx =$

4. $\int \frac{1}{x\sqrt{4x^2+16}}\,dx =$

5. $\int \frac{1}{\left(x^2+3\right)^{3/2}}\,dx =$

6. $\int e^x\sqrt{1-e^{2x}}\,dx =$

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Use Trigonometric Substitution if possible.

1. $\int \frac{x}{\sqrt{9-x^2}}\,dx =$

In the triangle: hyp = 3, opp = x adj = $\sqrt{9-x^2}$

$\sin\theta = \frac{x}{3} \rightarrow x = 3\sin\theta \rightarrow dx = 3\cos\theta\, d\theta \rightarrow \cos\theta = \frac{\sqrt{9-x^2}}{3} \rightarrow \sqrt{9-x^2} = 3\cos\theta$

$\int \frac{x}{\sqrt{9-x^2}}\,dx = \int \frac{3\sin\theta \bullet 3\cos\theta}{3\cos\theta}\,d\theta = 3\int \sin\theta\, d\theta = -3\cos\theta + C = -3\frac{\sqrt{9-x^2}}{3} + C = \boxed{-\sqrt{9-x^2} + C}$

2. $\int \sqrt{16-4x^2}\,dx =$

In the triangle: hyp = 4, opp = 2x, adj = $\sqrt{16-4x^2}$

$\sin\theta = \frac{1}{2}x \rightarrow x = 2\sin\theta \rightarrow dx = 2\cos\theta\, d\theta \rightarrow \cos\theta = \frac{\sqrt{16-4x^2}}{4} \rightarrow \sqrt{16-4x^2} = 4\cos\theta$

$\int \sqrt{16-4x^2}\,dx = \int 4\cos\theta \bullet 2\cos\theta\, d\theta = 8\int \cos^2\theta\, d\theta = 4\int (1+\cos 2\theta)\,d\theta = 4\theta + 2\sin 2\theta + C$

$= \boxed{4\arcsin\frac{x}{2} + x\sqrt{4-x^2} + C}$

3. $\int \frac{1}{\sqrt{x^2-9}}\,dx =$

In the triangle: hyp = x^2, adj = 3, opp = $\sqrt{x^2-9}$

$x = 3\sec\theta \rightarrow dx = 3\sec\theta\tan\theta\, d\theta \rightarrow \cos\theta = \frac{3}{\sqrt{x^2-9}} \rightarrow \frac{1}{\sqrt{x^2-9}} = \frac{1}{3}\cos\theta$

$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \sec\theta\, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C_1 = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C_1$

$= \boxed{\ln\left|x + \sqrt{x^2-9}\right| + C}$

4. $\int \frac{1}{x\sqrt{4x^2+16}}\,dx =$

In the triangle: opp = 2x, adj = 4, hyp = $\sqrt{4x^2+16}$

$\tan\theta = \frac{1}{2}x \rightarrow x = 2\tan\theta \rightarrow dx = 2\sec^2\theta\ d\theta \rightarrow \cos\theta = \frac{4}{\sqrt{4x^2+16}} \rightarrow \frac{1}{\sqrt{4x^2+16}} = \frac{1}{4}\cos\theta$

$\int \frac{1}{x\sqrt{4x^2+16}}\,dx = \int \frac{1}{2\tan\theta} \bullet \frac{1}{4}\cos\theta \bullet 2\sec^2\theta\, d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin\theta} \bullet \cos\theta \bullet \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{4}\int \csc\theta\, d\theta$

$= -\frac{1}{4}\ln\left|\csc\theta + \cot\theta\right| + C = -\frac{1}{4}\ln\left|\frac{\sqrt{x^2+4}}{x} + \frac{2}{x}\right| + C = \boxed{-\frac{1}{4}\ln\left|\frac{\sqrt{x^2+4}+2}{x}\right| + C}$

5. $\int \frac{1}{\left(x^2+3\right)^{3/2}}\,dx =$

In the triangle: opp = x, adj = $\sqrt{3}$, hyp = $\sqrt{x^2+3}$

$\cos\theta = \frac{\sqrt{3}}{\sqrt{x^2+3}} \rightarrow \frac{1}{\left(x^2+3\right)^{1/2}} = \frac{1}{\sqrt{3}}\cos\theta \rightarrow \frac{1}{\left(x^2+3\right)^{3/2}} = \frac{1}{3\sqrt{3}}\cos^3\theta \rightarrow \tan\theta = \frac{x}{\sqrt{3}} \rightarrow$

$x = \sqrt{3}\tan\theta \rightarrow dx = \sqrt{3}\sec^2\theta\, d\theta$

$\int \frac{1}{\left(x^2+3\right)^{3/2}}\,dx = \int \frac{1}{3\sqrt{3}}\cos^3\theta \bullet \sqrt{3}\sec^2\theta\, d\theta = \frac{1}{3}\int \cos\theta\, d\theta = \frac{1}{3}\sin\theta + C$

$= \boxed{\frac{1}{3}\cdot\frac{x}{\sqrt{x^2+3}} + C}$