The Rational Root Theorem

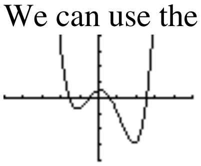
The solutions of the equation $64x^3 + 152x^2 - 62x - 105 = 0$ are $-\frac{5}{2}, -\frac{3}{4,}$ & $\frac{7}{8}$

Notice that the numerators (5, 3, & 7) are factors of 105 and the denominators (2, 4, & 8) are factors of 64. Generally if $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$ has integer coefficients, then every rational solution of f(x) has the following form: $\frac{p}{q} = \frac{factor\ of\ a_0}{factor\ of\ a_n}$. NOTE: This Theorem DOES NOT give the solutions, only possible solutions.

Example 1:
$f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$ Find all real zeros.
Solution:

The Rational Candidates are: $x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}$

We can use the graphing calculator to get an idea of the best choices.



The x-intercepts (zeros) are: in [-2, -1], in [-1, 0], in [0, 1], and in [3, 4].

Try $-\frac{3}{2}$ or $-1\frac{1}{2}$

-1 1/2	10	-11	-42	7	12
		-15	39	4 1/2	-17 1/4
	10	-26	-3	11 1/2	-5 1/4

$-\frac{3}{2}$ is not a zero

Try $-\frac{1}{2}$

- 1/2	10	-11	-42	7	12
		-5	8	17	-12
	10	-16	-34	24	0

$-\frac{1}{2}$ is a zero

So far we have $f(x) = (x + ½)(10x^3 - 16x^2 - 34x + 24)$
For $10x^3 - 16x^2 - 34x + 24$, we will try to find zeros.

Try $\frac{3}{5}$

3/5	10	-16	-34	24
		6	-6	-24
	10	-10	-40	0

$\frac{3}{5}$ is a zero.

So now we have: $f(x) = (x + ½)(x - 3/5)(10x^2 - 10x - 40)$
In the last parentheses we can divide by 2 but multiply the first parentheses by 2
Also in the last parentheses we can divide by 5 but multiply the 2nd parentheses by 5
$(2x + 1)(5x - 3)(x^2 - x - 4)$ Since $x^2 - x - 8$ cannot be factored, so we must use the quadratic formula to find the remaining zeros.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2} = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

Now, we have all the zeros: $\boxed{-\frac{1}{2}, \frac{3}{5}, \frac{1+\sqrt{17}}{2}, \ \& \ \frac{1-\sqrt{17}}{2}}$

Example 2:

$f(x) = x^3 - 8x^2 + 11x + 20$ Find all real roots (zeros).

Solution:

The Rational Candidates are: $x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}, \pm\frac{5}{1}, \pm, \frac{10}{1}, \pm\frac{20}{1} = \pm 1, \pm 2, \pm 4, \pm 5, \pm, 10, \pm 20$

Try 1

1	1	-8	11	20
		1	-7	4
	1	-7	4	24

1 did not give a zero.

Try -1

-1	1	-8	11	20
		-1	9	-20
	1	-9	20	0

-1 gave a zero.

So far we have the factorization: $f(x) = (x + 1)(x^2 - 9x + 20)$

To finish the factorization: $f(x) = (x + 1)(x + 4)(x - 5)$

The zeros are: $\boxed{-1, -4, \& -5}$

The Fundamental Theorem of Algebra:

If f(x) is a polynomial of degree n where n > 0, then the equation f(x) = 0 has at least one solution in the set of complex numbers.

Corollary to the Fundamental Theorem of Algebra:

If f(x) is a polynomial of degree n where n > 0, then the equation f(x) = 0 has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

The Complex Conjugates Theorem:

If f is a polynomial function with real coefficients, and a + bi is an imaginary zero of f, then a – bi is also a zero of f.

Example 3:

$f(x) = x^5 + x^3 - 2x^2 - 12x - 8$ Find all roots, real and imaginary.

Solution:

Try -1

-1	1	0	1	-2	-12	-8
		-1	1	-2	4	8
	1	-1	2	-4	-8	0

$(x + 1)(x^4 - x^3 + 2x^2 - 4x - 8)$

Try -1

-1	1	-1	2	-4	-8
		-1	2	-4	8
	1	-2	4	-8	0

$(x + 1)(x + 1)(x^3 - 2x^2 + 4x - 8)$

Try 2

2	1	-2	4	-8
		2	0	8
	1	0	4	0

$\boxed{(x + 1)(x + 1)(x - 2)(x^2 + 4) \quad \text{Zeros: } -1, -1, 2, \pm 2i}$.

Assignment 142

Page 202, #'s 10, 12, 14